Indian Statistical Institute, Bangalore Centre
M.Math. (I Year) : 2013-2014
Semester I : Mid-Semestral Examination
Measure theoretic Probability

16.09.2013 Time: $2\frac{1}{2}$ hours. Maximum Marks : 80

Note: Notation and terminology are understood to be as used in class. State clearly the results you are using in your answers.

1. (15 marks) Let $(\Omega, \mathcal{F}, \mu)$ be a Boolean measure space; let μ^* denote the corresponding outer measure. Show that μ^* is countably subadditive.

2. (15 marks) Let L^* denote the Lebesgue outer measure on the class of all subsets of $\mathbb{R}$. Show that $L^*(A + a) = L^*(A)$ for all $A \subseteq \mathbb{R}$, $a \in \mathbb{R}$.

3. (15+10 = 25 marks) (i) Let $(\Omega, \mathcal{B}, \mu)$ be a measure space with $\mu(\Omega) < \infty$. Let $\{f_n(\cdot)\}$ be a sequence of real valued integrable functions such that $\lim_{n\to\infty} f_n(\omega) = f(\omega)$ uniformly over $\omega \in \Omega$, for some function $f(\cdot)$. Show that $f(\cdot)$ is integrable w.r.t. μ, and that

$$\int_\Omega f(\omega)d\mu(\omega) = \lim_{n\to\infty} \int_\Omega f_n(\omega)d\mu(\omega).$$

(ii) Can the hypothesis $\mu(\Omega) < \infty$ in (i) be dropped?

4. (15+10 = 25 marks) (i) Let $(\Omega, \mathcal{B}, \mu)$ be a σ-finite measure space. Let λ be a totally finite measure on $(\Omega, \mathcal{B})$ such that $\lambda(E) = \int_E f(\omega)d\mu(\omega)$ for $E \in \mathcal{B}$, where f is a nonnegative measurable function. Let g be a real valued measurable function on Ω. Show that g is integrable w.r.t. λ if and only if $(g.f)$ is integrable w.r.t. μ, and in such a case

$$\int_\Omega g(\omega)d\lambda(\omega) = \int_\Omega g(\omega)f(\omega)d\mu(\omega).$$

(ii) Let X be a real valued random variable on a probability space $(\Omega, \mathcal{B}, P)$ with probability density function $f_X(\cdot)$. Show that $E(X)$ exists if and only if $\int_\mathbb{R} |x| f_X(x)dx < \infty$, and in such a case $E(X) = \int_\mathbb{R} x f_X(x)dx < \infty$.